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Lagrange and Hermite Interpolation in Banach Spaces*

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Let f map a Banach space X into itself, and let $x_1, x_2, \ldots, x_n$ be distinct points of X. Then there exists a polynomial $y(x)$ of degree $(n-1)$ which interpolates f at these points. Furthermore, $y(x)$ has a Lagrange representation

$$y(x) = \sum_{i=1}^{n} [w_i'(x_i)]^{-1} \frac{w_i(x)}{(x - x_i)} f(x_i),$$

where $w_i(x) = L_i(x - x_1, x - x_2, \ldots, x - x_n)$, $w_i'(x_i)$ is the first Fréchet derivative of w_i at x_i, and L_i, $i = 1, 2, \ldots, n$, is an appropriately chosen n-linear operator. In an analogous manner, an Hermite polynomial $\bar{y}(x)$ of degree $(2n-1)$ is derived, which interpolates f and f' at $x_1, x_2, \ldots, x_n$. Finally, if X is a Hilbert space, the polynomials $y(x)$ and $\bar{y}(x)$ are shown to have simple representations in terms of inner products.

1. INTRODUCTION

Let X and Y be Banach spaces and let f be a function mapping X into Y. If X and Y are the real line R^1, the classical Lagrange and Hermite interpolation problems are, respectively, to find a polynomial $y(x)$ of degree $(n-1)$ which interpolates f at n given distinct points $x_1, x_2, \ldots, x_n$, and to find a polynomial $\bar{y}(x)$ of degree $(2n-1)$ which interpolates f at $x_1, x_2, \ldots, x_n$ while $\bar{y}'$ interpolates f' at these points. In this paper we solve the Banach space analogs of these two problems, using polynomial operators. That is, we exhibit polynomials $y(x)$ and $\bar{y}(x)$, of degrees $n-1$ and $2n-1$, such that y interpolates f at the n given distinct points $x_1, x_2, \ldots, x_n$, $\bar{y}$ interpolates f, and $\bar{y}'$ interpolates f' at these points. In particular, we show that $y(x)$ has a Lagrange representation

$$y(x) = \sum_{i=1}^{n} [w_i'(x_i)]^{-1} \frac{w_i(x)}{(x - x_i)} f(x_i), \tag{1}$$

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where $w_i'(x_i)$ is the Fréchet derivative of w_i at x_i, and $w_i(x)$ is the n-th degree polynomial

$$w_i(x) = L_i(x - x_1, x - x_2, ..., x - x_n),$$

L_i being an appropriately chosen n-linear operator. In the event X is a Hilbert space, the polynomials $y(x)$ and $\bar{y}(x)$ are shown to have simple representations in terms of inner products.

2. Polynomials in a Linear Space

Let X be a linear space over the field of real (complex) numbers. For each $k = 1, 2, ...,$ let X^k denote the direct product

$$\underbrace{X \times X \times \cdots \times X.}_{k \text{ times}}$$

A *k-linear operator M on X*, is a function on X^k into a linear space Y which is linear and homogeneous in each of its arguments separately. That is, for each $i = 1, 2, ..., k$,

$$M(x_1, x_2, ..., x_i + y_i, ..., x_k)$$
$$= M(x_1, x_2, ..., x_i, ..., x_k) + M(x_1, x_2, ..., y_i, ..., y_k),$$

and

$$M(x_1, x_2, ..., ax_i, ..., x_n) = aM(x_1, x_2, ..., x_i, ..., x_n).$$

A 0-*linear operator* L_0, *on X*, is a constant function. That is, for some fixed $y \in Y$, $L_0 x = y$ for all $x \in X$. We shall identify a 0-linear operator L_0 with its range so that $L_0 x = L_0$ for all $x \in X$. In the event $x_1 = x_2 = \cdots = x_k = x$ we shall adopt the notation

$$M(x_1, x_2, ..., x_k) = Mx^k,$$

where M is a k-linear operator.

For $k = 0, 1, 2, ..., n$, let L_k be a k-linear operator on X. Then the operator P on X into Y given by

$$P(x) = L_0 + L_1 x + L_2 x^2 + \cdots + L_n x^n$$

is called a *polynomial of degree n on X*.

Let $\mathscr{L}_n[X, Y]$, $n = 0, 1, 2, ...,$ denote the set of n-linear operators on X into Y. If $X = Y$, we shall simply write $\mathscr{L}_n[X]$; we shall identify $\mathscr{L}_0[X]$ with X.

If $L \in \mathscr{L}_n[X, Y]$, $n > 1$, then for each $x \in X$, $L(x) \in \mathscr{L}_{n-1}[X, Y]$ is the $(n-1)$-linear operator defined by

$$(L(x))(x_2, x_3, \ldots, x_n) = L(x, x_2, x_3, \ldots, x_n).$$

In general, if $n > k \geqslant 1$, then for each

$$x_1, x_2, \ldots, x_k \in X, \qquad L(x_1, x_2, \ldots, x_k) \in \mathscr{L}_{n-k}[X, Y]$$

is the $(n-k)$-linear operator defined by

$$(L(x_1, x_2, \ldots, x_k))(x_{k+1}, \ldots, x_n) = L(x_1, x_2, \ldots, x_n).$$

If L is n-linear $(n > 1)$, we shall let $\partial_i L$ denote the $(n-1)$-linear operator on X into $\mathscr{L}_1[X, Y]$ defined by

$$\partial_i L(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = L(x_1, x_2, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n),$$

where

$$(L(x_1, x_2, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n))(x) = L(x_1, x_2, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n).$$

In general, n-linear operators are not *symmetric*. That is, it need not be true that

$$L(x_1, x_2, \ldots, x_n) = L(x_{i_1}, x_{i_2}, \ldots, x_{i_n})$$

for all permutations $(i_1, i_2, \ldots, i_n)$ of $(1, 2, \ldots, n)$. For this reason, in general,

$$\partial_i L(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \neq L(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

Now let L be any n-linear operator and let $x_1, x_2, \ldots, x_n$ be any points of X. We define a function w on X into Y by

$$w(x) = L(x - x_1, x - x_2, \ldots, x - x_n).$$

Clearly, $w(x)$ is a polynomial

$$L_n x^n + L_{n-1} x^{n-1} + \cdots + L_1 x + L_0$$

of degree n on X, where $L_n = L$, and $L_0 = (-1)^n L(x_1, x_2, \ldots, x_n)$. For example, if L is bilinear,

$$L(x - x_1, x - x_2) = Lx^2 - L(x_1, x) - L(x, x_1) + L(x_1, x_2).$$

Thus $L_2 = L$, $L_0 = L(x_1, x_2)$, and $L_1 = -L(x_1, \cdot) - L(\cdot, x_2)$.

An n-linear operator L is said to be *bounded* provided there exists a constant $M > 0$ for which

$$\| L(x_1, x_2, \ldots, x_n) \| \leqslant M \| x_1 \| \cdot \| x_2 \| \cdot \cdots \cdot \| x_n \|.$$

Analogously to the 1-linear case, it can be proved that an n-linear operator L is bounded if and only if it is continuous. Continuity of L is defined in terms of the product topology on X^n. If we define

$$\| L \| = \inf\{M : \| L(x_1, x_2, ..., x_n)\| \leqslant M \| x_1 \| \cdot \| x_2 \| \cdot \cdots \cdot \| x_n \|\},$$

then

$$\| L \| = \sup\{\| L(x_1, x_2, ..., x_n) : \| x_i \| = 1, i = 1, 2, ..., n\}. \tag{1}$$

Clearly, whenever Y is a Banach space, $\mathscr{L}_n[X, Y]$, with the norm (1), is also a Banach space.

Finally, it will be useful to note [3] that $\mathscr{L}_n[X, Y]$ is isometric to $\mathscr{L}_1[X, \mathscr{L}_{n-1}[X]]$, which is isometric to $\mathscr{L}_1[X, \mathscr{L}_1[X, \mathscr{L}_1[X, ..., \mathscr{L}_1[X, Y] \cdots]]]$.

3. Fréchet Derivatives of Operators

Let f be a function mapping an open subset V of a Banach space X into a Banach space Y. Let $x_0 \in V$. If there exists a linear operator $U \in \mathscr{L}_1[X, Y]$ such that

$$\| f(x_0 + \Delta x) - f(x_0) - U(\Delta x) \| = o(\| \Delta x \|),$$

then $U = f'(x_0)$ is called the *Fréchet derivative of f at x_0*. Equivalently,

$$U(x) = \lim_{t \to 0} \frac{f(x_0 + tx) - f(x_0)}{t},$$

where the convergence is uniform on the sphere $\{x : \| x \| = 1\}$. It follows from this definition that if L is a bounded, n-linear operator on X, and $f(x) = Lx^n$, then $f'(x) = \sum_{i=1}^{n} \partial_i L x^{n-1}$. In particular, if L is bilinear and $f(x) = Lx^2$, then $f'(x) = L(x, \cdot) + L(\cdot, x)$. If L is symmetric, then, clearly, $f'(x) = nLx^{n-1}$.

We shall need the derivative of w. Let L be n-linear and let $x_1, x_2, ..., x_n$ be points of X. We let $\partial_i w$ or $w/(x - x_i)$ denote the operator on X into $\mathscr{L}_i[X, Y]$ defined by

$$\partial_i w(z) = L(z - x_1, z - x_2, ..., z - x_{i-1}, \cdot, z - x_{i+1}, ..., z - x_n).$$

We set

$$\partial_i w(z) = (w/(x - x_i))(z) = w(z)/(x - x_i).$$

It should be noted that the operator $w/(x - x_i)$ is completely independent of the x in the denominator; the denominator $(x - x_i)$ is purely symbolic.

THEOREM 3.1. *Let L be a bounded, n-linear operator. Let $x_1, x_2, \ldots, x_n \in X$, and set*

$$w(x) = L(x - x_1, x - x_2, \ldots, x - x_n).$$

Then $w'(x_0) = \sum_{i=1}^{n} w(x_0)/(x - x_i)$ and, in particular, $w'(x_i) = w(x_i)/(x - x_i) = \partial_i w(x_i)$.

Proof. Let x_0 be a fixed point of X. Then, using the multilinearity and boundedness of L,

$$\left\| w(x_0 + \Delta x) - w(x_0) - \sum_{i=1}^{n} \frac{w(x_0)}{(x - x_i)} (\Delta x) \right\|$$

$$= \Big\| L(x_0 - x_1 + \Delta x, x_0 - x_2 + \Delta x, \ldots, x_0 - x_n + \Delta x)$$

$$- L(x_0 - x_1, \ldots, x_0 - x_n)$$

$$- \sum_{i=1}^{n} L(x_0 - x_1, \ldots, x_0 - x_{i-1}, \Delta x, x_0 - x_{i+1}, \ldots, x_0 - x_n) \Big\|$$

$$\leqslant \sum_{k=2}^{n} M_k \| \Delta x \|^k = o(\| \Delta x \|),$$

where each M_k is a positive constant arising from $\| L \|$ and from the norms $\| x_0 - x_i \|$, $i = 1, 2, \ldots, n$.

One can speak also of higher order Fréchet derivatives. If $f: X \to Y$ and if f' exists on an open neighborhood V of x_0 in X, then $f''(x_0) = (f')'(x_0)$ is a linear operator on X into $\mathscr{L}_1[X, Y]$ for which

$$\| f'(x_0 + \Delta x) - f'(x_0) - f''(x_0)(\Delta x) \| = o(\| \Delta x \|).$$

Thus, $f''(x_0) \in \mathscr{L}_1[X, \mathscr{L}_1[X, Y]]$ and, since $\mathscr{L}_1[X, \mathscr{L}_1[X, Y]]$ is isometric to $\mathscr{L}_2[X, Y]$, it follows that $f''(x_0)$ can be considered a bilinear operator on X into Y which is usually not symmetric. In general, $f^{(n)}(x_0)$, the n-th Fréchet derivative of f at x_0, is a linear operator on X into $\mathscr{L}_{n-1}[X, Y]$; so that $f^{(n)}(x_0)$ can be considered as belonging to $\mathscr{L}_n[X, Y]$.

Some examples of Fréchet derivatives are instructive. Let $X = Y = R^n$, the (real) Euclidean n-space. Then, if $f: X \to Y$ and for $(x_1, x_2, \ldots, x_n) \in X$, $f(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n)$, where $y_i = f_i(x_1, x_2, \ldots, x_n)$, $i = 1, 2, \ldots, n$, each f_i is a real-valued function of n real variables. It can then be shown that if each f_i has continuous first partial derivatives on some open set V in X, then $f'(x)$ exists on V and is given by the matrix

$$f'(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \frac{\partial f_2}{\partial x_1}(x) & \frac{\partial f_2}{\partial x_2}(x) & \cdots & \frac{\partial f_2}{\partial x_n}(x) \\ \vdots & & & \\ \frac{\partial f_n}{\partial x_1}(x) & & \cdots & \frac{\partial f_n}{\partial x_n}(x) \end{bmatrix}$$

$$= \left(\frac{\partial f_i}{\partial x_j}(x)\right), \qquad i, j = 1, 2,..., n,$$

which is the gradient of f at x. Analogously, if each f_i has continuous second partials on some neighborhood U of x, then $f''(x)$ is given by the three-way matrix

$$f''(x) = \frac{\partial^2 f_i(x)}{\partial x_j \, \partial x_k}, \qquad i, j, k = 1, 2,..., n,$$

which is the Hessian of f at x.

More generally, one can show that if L is a bounded, n-linear operator on X, then $L^{(k)}(x)$ is a bounded, k-linear operator on X.

4. The Interpolation Problem — Existence

Let $c_1, c_2,..., c_n$ be points of a Banach space X. The *interpolation problem* is that of finding, for each sequence $\{x_1, x_2,..., x_n\}$ of distinct points of X, a polynomial operator p which interpolates $\{c_1, c_2,..., c_n\}$ at $\{x_1, x_2,..., x_n\}$, so that $p(x_i) = c_i$. We shall prove that there always exists a polynomial of degree $(n - 1)$ which solves the interpolation problem.

To this end, let L be a bounded n-linear operator in $\mathscr{L}_n[X]$; let $x_1, x_2,..., x_n$ be distinct points of X and let $w(x) = L(x - x_1, x - x_2,..., x - x_n)$. Then w is a polynomial of degree n mapping X into X, and

$$\frac{w(x)}{(x - x_i)} = \partial_i w(x) = L(x - x_1, x - x_2,..., x - x_{i-1}, x - x_{i+1},..., x - x_n),$$

is a polynomial of degree $(n - 1)$ which maps X into $\mathscr{L}_1[X]$. We have shown that $w'(x) = \sum_{i=1}^n w(x)/(x - x_i)$, so that $w'(x_i) = w(x_i)/(x - x_i) = \partial_i w(x_i)$ is a linear operator. Thus, should $w'(x_i)$ be nonsingular for $i = 1, 2,..., n$, then since $l_i(x) = [w'(x_i)]^{-1} w(x)/(x - x_j)$, l_i would be a linear and operator-valued function having the property

$$l_i(x_j) = \delta_{ij} I.$$

Furthermore, for each $x_0 \in X$, it is easily seen that $[l_i(x)](x_0) = l_i(x)\, x_0$ is a polynomial of degree $(n - 1)$. That is, we have proved

THEOREM 4.1. *If there exists an n-linear operator L such that* $[w'(x_i)]^{-1}$ *exists for each* $i = 1, 2,..., n$, *where*

$$w(x) = L(x - x_1, x - x_2, ..., x - x_n),$$

then the Lagrange polynomial $y(x)$ *of degree* $(n - 1)$ *given by*

$$y(x) = \sum_{i=1}^{n} l_i(x)\, c_i \left(= \sum_{i=1}^{n} l_i(x) f(x_i) \right),$$

where $l_i(x) = [w'(x_i)]^{-1} w(x)/(x - x_i) = [w'(x_i)]^{-1} \partial_i w(x)$, *solves the interpolation problem (interpolates the function f at the n distinct points* $x_1, x_2, ..., x_n$ *of X).*

Thus, to solve the interpolation problem, it is enough to prove that such an n-linear operator exists. It would actually suffice to prove the existence of a family $\{L_1, L_2, ..., L_n\}$ of n-linear operators having the property that $[w_i'(x_i)]^{-1}$ exists for $i = 1, 2,..., n$, where $w_i(x) = L_i(x - x_1, x - x_2, ..., x - x_n)$. If this were the case, we could take

$$y(x) = \sum_{i=1}^{n} [w_i'(x_i)]^{-1} \frac{w_i(x)}{(x - x_i)} (c_i)$$

as our interpolating polynomial. We shall prove the existence of such a family of L_i's.

THEOREM 4.2. *Let* $x_1, x_2, ..., x_n$ *be distinct points of a Banach space X. Then for each* $i = 1, 2,..., n$ *there exists an n-linear operator* L_i *for which* $[w_i'(x_i)]^{-1}$ *exists, where*

$$w_i(x) = L_i(x - x_1, x - x_2, ..., x - x_n).$$

Furthermore, the L_i's *can be chosen so that* $w_i'(x_i) = I$, *where I is the identity operator in* $\mathscr{L}_1[X]$.

Proof. We start with $i = 1$. We must produce an n-linear operator L_1 for which $w_1'(x_1)$ exists and is nonsingular, where

$$w_1(x) = L_1(x - x_1, x - x_2, ..., x - x_n).$$

Recall that if such an L_1 exists, then

$$w_1'(x_1) = \frac{w_1(x_1)}{(x - x_1)} = \partial_1 w_1(x_1)$$
$$= L_1(\cdot, x_1 - x_2, x_1 - x_3, ..., x_1 - x_n),$$

which belongs to $\mathscr{L}_1[X]$. Also, $L_1 : X^{n-1} \to \mathscr{L}_1[X]$. With this in mind, let $X_{1j} = \operatorname{span}\{x_1 - x_j\}$. Since each X_{1j} $(j = 2, 3, ..., n)$ is one-dimensional, there exist continuous projections P_{1j} of X onto X_{1j}. Define

$$\tilde{T}_1 : X_{12} \times X_{13} \times \cdots \times X_{1n} \to \mathscr{L}_1[X]$$

by linearity, through the equation

$$\tilde{T}_1(x_1 - x_2, x_1 - x_3, ..., x_1 - x_n) = I.$$

Then $\tilde{T}_1$ is a bounded (continuous), $(n - 1)$-linear operator in

$$\mathscr{L}_1[X_{12} \times X_{13} \times \cdots \times X_{1n}, Y].$$

That is,

$$\begin{aligned}
&\| \tilde{T}_1(a_2(x_1 - x_2), a_3(x_1 - x_3), ..., a_n(x_1 - x_n))\| \\
&\quad = \| a_2 a_3 \cdots a_n \tilde{T}_1(x_1 - x_2, x_1 - x_3, ..., x_1 - x_n)\| \\
&\quad = | a_2 a_3 \cdots a_n | \cdot \| I \| \\
&\quad = \frac{1}{\| x_1 - x_2 \| \| x_1 - x_3 \| \cdots \| x_1 - x_n \|} \| a_1(x_1 - x_2) \| \cdots \| a_n(x_1 - x_n) \|,
\end{aligned}$$

so that $\| \tilde{T}_1 \| = 1/\| x_1 - x_2 \| \| x_1 - x_3 \| \cdots \| x_1 - x_n \|$.

We extend $\tilde{T}_1$ to a continuous, $(n - 1)$-linear operator $T_1 : X^{n-1} \to \mathscr{L}_1[X]$ through the projections P_{1j}. That is, we define

$$T_1(y_1, y_2, ..., y_{n-1}) = \tilde{T}_1(P_{12} y_1, P_{13} y_2, ..., P_{1n} y_{n-1}).$$

Since the projections P_{1j} are linear and continuous, it follows that T_1 is $(n - 1)$-linear and continuous. In particular, the map P,

$$P : X^{n-1} \to X_{12} \times X_{13} \times \cdots \times X_{1n}$$

given by $P(y_2, y_3, ..., y_n) = (P_{12} y_2, P_{13} y_3, ..., P_{1n} y_n)$ is continuous, so that the composition $\tilde{T}_1 \circ P = T_1$, is continuous.

Now define the n-linear operator L_1 by

$$L_1(y_1, y_2, ..., y_n) = [T_1(y_2, y_3, ..., y_n)](y_1).$$

The n-linearity of L_1 follows directly from the $(n-1)$-linearity of T_1 and the fact that T_1 is linear and operator-valued. The boundedness of T_1 is also apparent. If $P_{1k}y_k = a_k[(x_1 - x_k)/\| x_1 - x_k \|]$, then $\| P_{1k}y_k \| = | a_k |$. Thus,

$$\begin{aligned} L_1(y_1, y_2, \dots, y_{n-1}, y_n) &= [T_1(y_2, y_3, \dots, y_n)](y_1) = [\tilde{T}_1(P_{12}y_2, P_{13}y_3, \dots, P_{1n}y_n)](y_1) \\ &= \frac{a_2 \cdot a_3 \cdots a_n}{\| x_1 - x_2 \| \cdot \| x_1 - x_3 \| \cdots \| x_1 - x_n \|} [\tilde{T}_1(x_1 - x_2, \dots, x_1 - x_n)](y_1) \\ &= \frac{a_2 \cdot a_3 \cdots a_n}{\| x_1 - x_2 \| \cdot \| x_1 - x_3 \| \cdots \| x_1 - x_n \|} y_1 . \end{aligned}$$

Therefore, if $K = 1/\| x_1 - x_2 \| \cdot \| x_1 - x_3 \| \cdots \| x_1 - x_n \|$, then

$$\begin{aligned} \| L_1(y_1, y_2, \dots, y_n) \| &= K | a_1 | \cdot | a_2 | \cdots | a_n | \, \| y_1 \| \\ &= K \cdot \| P_{12}y_2 \| \cdot \| P_{13}y_3 \| \cdots \| P_{1n}y_n \| \cdot \| y_1 \| \\ &\leqslant \bar{K} \| y_1 \| \cdot \| y_2 \| \cdots \| y_n \|, \end{aligned}$$

since each P_{1k} is a projection and $\| P_{1k}y \| = \| P_{1k} \| \cdot \| y \|$.

Now let $w_1(x) = L_1(x - x_1, x - x_2, \dots, x - x_n)$. Since L_1 is a bounded, n-linear operator, $w_1(x)$ is differentiable and

$$\begin{aligned} w_1'(x_1) &= \frac{w(x_1)}{(x - x_1)} \\ &= L_1(\cdot, x_1 - x_2, x_1 - x_3, \dots, x_1 - x_n) \\ &= \tilde{T}_1(x_1 - x_2, x_1 - x_3, \dots, x_1 - x_n) \\ &= I \end{aligned}$$

Thus $w_1'(x_1)$ is a non-singular, linear operator.

A similar line of argument proves the existence, for each $i = 1, 2, \dots, n$, of an n-linear operator L_i for which $w_i'(x_i) = I$, where

$$w_i(x) = L_i(x - x_1, x - x_2, \dots, x - x_n).$$

This completes the proof of the theorem.

As a direct result of Theorem 4.2 we have

THEOREM 4.3. *The interpolation problem can always be solved by a polynomial $y(x)$ of degree $(n-1)$ having a Lagrange representation*

$$y(x) = \sum_{i=1}^{n} l_i(x) \, c_i ,$$

where $l_i(x) = [w_i'(x_i)]^{-1} w_i(x)/(x - x_i) = [w_i'(x_i)]^{-1} \partial_i w_i(x)$ *and* $w_i(x) = L_i(x - x_1, x - x_2, \ldots, x - x_n)$ *for appropriately chosen n-linear operators* $L_1, L_2, \ldots, L_n$.

In the event X is a Hilbert space with inner product (x, y), Theorem 4.2 also yields a representation theorem. Consider the projection P_{1j} of X onto X_{1j} given in the proof of Theorem 4.2. If X is a Hilbert space, then

$$P_{1j} y_j = \left(y_j, \frac{x_1 - x_j}{\| x_1 - x_j \|}\right) \frac{x_1 - x_j}{\| x_1 - x_j \|}.$$

Thus

$$L_1(y_1, y_2, \ldots, y_n) = \frac{(y_2, x_1 - x_2) \cdot (y_3, x_1 - x_3) \cdots (y_n, x_1 - x_n)}{\| x_1 - x_2 \|^2 \cdot \| x_1 - x_3 \|^2 \cdots \| x_1 - x_n \|^2} I(y_1).$$

In particular, since $w_1'(x_1) = I$,

$$\begin{aligned} l_1(x) &= I \circ \frac{w_1(x)}{(x - x_1)} \\ &= L_1(\cdot, x - x_2, x - x_3, \ldots, x - x_n) \\ &= \frac{(x - x_2, x_1 - x_2) \cdot (x - x_3, x_1 - x_3) \cdots (x - x_n, x_1 - x_n)}{\| x_1 - x_2 \|^2 \cdot \| x_1 - x_3 \|^2 \cdots \| x_1 - x_n \|^2} I. \end{aligned}$$

Analogously, one can prove that

$$l_j(x) = \left[\prod_{\substack{k=1 \\ k \neq j}}^{n} (x - x_k, x_j - x_k)\right] \left[\prod_{\substack{k=1 \\ k \neq j}}^{n} \| x_j - x_k \|\right]^{-1} I.$$

Thus we arrive at

THEOREM 4.4. *Let X be a Hilbert space with inner product (x, y) and let $c_1, c_2, \ldots, c_n$ be points of X. Then, for any distinct points $x_1, x_2, \ldots, x_n$ of X, the polynomial $y(x)$ of degree $n - 1$, given by*

$$y(x) = \sum_{i=1}^{n} \frac{\pi_i(x)}{\pi_i(x_i)} c_i,$$

where

$$\pi_i(x) = \prod_{\substack{k=1 \\ k \neq i}}^{n} (x - x_k, x_i - x_k),$$

satisfies $y(x_i) = c_i$, $i = 1, 2, \ldots, n$.

This theorem is evident by inspection; however, it is interesting to note how it followed naturally from the theory of Theorems 4.2 and 4.3.

5. Hermite Interpolation in Banach Spaces

Recall the classical Hermite polynomial $y(x)$ of degree $(2n - 1)$ which interpolates a real-valued function f of a real variable at the n distinct points $x_1, x_2, \ldots, x_n$ and for which $y'(x)$ interpolates f' at these points. This $y(x)$ is given by the formula

$$y(x) = \sum_{i=1}^{n} \{H_i(x) f(x_i) + \bar{H}_i(x) f'(x_i)\},$$

where $H_i(x) = [1 - 2l_i'(x_i)(x - x_i)]\, l_i^2(x)$, and $\bar{H}_i(x) = (x - x_i)\, l_i^2(x)$. Here $l_i(x)$ is the polynomial $w(x)/w'(x_i)(x - x_i)$ occurring in the classical Lagrange formula, and $w(x) = (x - x_1)(x - x_2) \cdots (x - x_n)$. It follows that

$$H_i(x_j) = \delta_{ij} = \bar{H}_i'(x_j),$$

and

$$H_i'(x_j) = 0 = \bar{H}_i(x_j), \qquad \text{for} \quad i, j = 1, 2, \ldots, n.$$

Now suppose X is a Banach space and f is a function from X into X which has a continuous Fréchet derivative at n distinct points $x_1, x_2, \ldots, x_n$ of X. Referring to Theorem 4.2, let $l_i(x) = [w_i'(x_i)]^{-1}\, w_i(x)/(x - x_i) = [w_i'(x_i)]^{-1} \partial_i w_i(x)$. Since $l_i(x)$ is linear and operator-valued, $l_1^2(x) = l_i(x) \circ l_i(x)$, being the composition of two linear operators, is itself linear and operator-valued. Furthermore, $l_i' : X \to \mathscr{L}_1[X, \mathscr{L}_1[X]]$ so that $[l_i'(x)](y)$ is linear and operator-valued. It is thus obvious that, for each $x \in X$, $l_i'(x_i)(x - x_i)$ is linear and operator-valued. We now define the Banach space analog of the above function $H_i(x)$ to be the linear operator-valued function on X:

$$H_i(x) = [I - 2l_i'(x_i)(x - x_i)]\, l_i^2(x),$$

where I is the identity in $\mathscr{L}_1[X]$. Since $l_i(x_j) = \delta_{ij} I$, it is evident that

$$H_i(x_j) = \delta_{ij} I. \tag{1}$$

Furthermore, we can show that $H_i'(x_j) = 0$, the zero linear operator from X to $\mathscr{L}_1[X]$, for $i, j = 1, 2, \ldots, n$. A proof of this requires some basic facts about Fréchet derivatives [5].

If A is a linear operator from X into Y, then $A'(x) = A$ for all $x \in X$. If $F : X \to Y$ and $F(x) = L_0$, a constant, for all $x \in X$, then $F'(x) = 0 \in \mathscr{L}_1[X, Y]$ for all $x \in X$. Let X, Y and Z be Banach spaces, and let $F : X \to Y$ and $G : Y \to Z$ be functions such that F is differentiable at x_0 and G is differentiable at $y_0 = F(x_0)$. Then GF is differentiable at x_0, and $(GF)'(x_0) = G'(y_0)\, F'(x_0)$. In particular, if G is linear, $(GF)'(x_0) = GF'(x_0)$. Finally, LEMMA 5.1. *Let A and B be functions from X into $\mathscr{L}_1[X]$ which are bounded, linear and operator-valued. If both A and B are differentiable at x_0 and if $F(x) = A(x)\, B(x)$, then*

$$F'(x_0)(x) = A(x_0)\, B'(x_0)(x) + A'(x_0)(x)\, B(x_0).$$

Proof. The proof follows directly from the continuity of A and B at x_0 and the definition of the Fréchet derivative.

Now let $A_i(x) = I - 2l_i'(x_i)(x - x_i)$, $B(x) = l_i^2(x)$. Then $A_i'(x_0) = -2l_i'(x_i)$ since I and $-2l_i'(x_i)(x_i)$ are constant and $l_i'(x_i)$ is a linear operator. Using Lemma 5.1 we see that

$$B_i'(x_j)(x) = l_i(x_j)\, l_i'(x_j)(x) + l_i'(x_j)(x)\, l_i(x_j)$$

so that

$$B_i'(x_j)(x) = \begin{cases} 0 \in \mathscr{L}_1[X] & \text{if } j \neq i, \\ -2l_i'(x_i)(x) & \text{if } j = i. \end{cases}$$

But $H_i(x) = A_i(x)\, B_i(x)$ so that, invoking again Lemma 5.1,

$$\begin{aligned} H_i'(x_j)(x) &= A_i'(x_j)(x)\, B_i(x_j) + A_i(x_j)\, B_i'(x_j)(x) \\ &= -2l_i'(x_i)(x)\, l_i^2(x_j) + [I - 2l_i'(x_i)(x_j - x_i)]\, B_i'(x_j)(x) \\ &= -2l_i'(x_i)(x)\, \delta_{ij} I \\ &\quad + [I - 2l_i'(x_i)(x_i - x_j)][(\delta_{ij} I)\, l_i'(x_i)(x) + l_i'(x_j)(x)(\delta_{ij} I)] \\ &= \begin{cases} 0 \in \mathscr{L}_1[X] & \text{if } j \neq i, \\ 2l_i'(x_i)(x) - 2l_i'(x_i)(x) = 0 \in \mathscr{L}_1[X] & \text{if } j = i. \end{cases} \end{aligned}$$

That is, $H_i'(x_j) = 0 \in \mathscr{L}_1[X, \mathscr{L}_1[X]]$ for all $i, j = 1, 2,..., n$.

If $\bar{H}_i(x)$ were a polynomial of degree $2n - 1$ from X into X for which $\bar{H}_i(x_j) = 0$ for all $i, j = 1, 2,..., n$, and for which $\bar{H}_i'(x_j) = \delta_{ij} I$, then

$$y(x) = \sum_{i=1}^{n} \{H_i(x)\, f(x_i) + f'(x_i)\, \bar{H}_i(x)\} \tag{2}$$

would be a polynomial of degree $2n - 1$ interpolating f at $x_1, x_2, ..., x_n$, with y' interpolating f' at these points. This follows directly from

$$y'(x) = \sum_{i=1}^{n} H_i'(x)\, f(x_i) + f'(x_i)\, \bar{H}_i'(x).$$

Note that, since $H_i(x) \in X$ and $f'(x_i) \in \mathscr{L}_1[X]$, $f'(x_i)$ must precede $H_i(x)$ in formula (2). Looking at the proof of Theorem 4.2, we find it can be readily adapted to produce a $(2n - 1)$-linear operator L_i, for each $i = 1, 2,..., n$, for which $[w_i'(x_i)]^{-1}$ exists and equals I, where

$$\begin{aligned} w_i(x) &= L_i(x - x_1, x - x_1, x - x_2, x - x_2,..., x - x_i,..., x - x_n, x - x_n) \\ &= L_i((x - x_1)^2, (x - x_2)^2,..., (x - x_i),..., (x - x_n)^2). \end{aligned}$$

It follows easily that

$$\bar{H}_i(x) = [w_i'(x_i)]^{-1} w_i(x) = w_i(x)$$

obeys the following relations:

$$\begin{aligned} \bar{H}_i(x_j) &= 0 \qquad \text{for all} \quad i, j = 1, 2,..., n, \\ \bar{H}_i'(x_j) &= \delta_{ij} I. \end{aligned}$$

Thus we arrive at

THEOREM 5.2. *Let $x_1, x_2,..., x_n$ be distinct points of a Banach space X and let $f: X \to X$ be differentiable at $x_1, x_2,..., x_n$. Then there exists a polynomial y of degree $(2n - 1)$,*

$$y(x) = \sum_{i=1}^{n} \{H_i(x) f(x_i) + f'(x_i) \bar{H}_i(x)\}, \tag{3}$$

which interpolates f at $x_1, x_2,..., x_n$, with $y'(x)$ interpolating f' at these points. Furthermore,

$$H_i(x) = [I - 2l_i'(x_i)(x - x_i)]\, l_i^2(x), \qquad \text{and} \qquad \bar{H}_i(x) = [w_i'(x_i)]^{-1} w_i(x),$$

where $w_i(x) = L_i((x - x_1)^2, (x - x_2)^2,..., (x - x_i),..., (x - x_n)^2)$, L_i being an appropriately chosen $(2n - 1)$-linear operator. I is the identity in $\mathscr{L}_1[X]$.

In the event X is a Hilbert space, we can obtain a simple representation of $y(x)$ in terms of inner products. First, one can show

$$\bar{H}_i(x) = \frac{\pi_i^2(x)}{\pi_i^2(x_i)} (x - x_i), \qquad \text{where} \quad \pi_i(x) = \prod_{\substack{k=1 \\ k \neq i}}^{n} (x - x_k, x_i - x_k)$$

and (,) denotes inner product. Then, since $l_i(x) = \pi_i(x)/\pi_i(x_i)\, I$, it follows upon differentiation that

$$l_i'(x)(y) = \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{\pi_i(x) \cdot (y, x - x_j)}{(x - x_j, x_i - x_j) \cdot \pi_i(x_i)} I.$$

Thus

$$l_i'(x_i)(x - x_i) = \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{(x - x_i, x_i - x_j)}{(x - x_j, x_i - x_j)} I$$

and

$$H_i(x) = \left[1 - \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{(x - x_i, x_i - x_j)}{(x - x_j, x_i - x_j)}\right] \frac{\pi_i^2(x)}{\pi_i^2(x_i)} I.$$

Therefore, we arrive at

THEOREM 5.3. *Let X be a Hilbert space with inner product (x, y) and let $x_1, x_2, \ldots, x_n$ be distinct points of X. Then the polynomial of degree $2n - 1$ given by*

$$y(x) = \sum_{i=1}^{n} \{H_i(x) f(x_i) + f'(x_i) \bar{H}_i(x)\},$$

where

$$H_i(x) = \left[1 - \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{(x - x_i, x_i - x_j)}{(x - x_j, x_i - x_j)}\right] \frac{\pi_i^2(x)}{\pi_i^2(x_i)} I$$

and

$$\bar{H}_i(x) = \left[\frac{\pi_i^2(x)}{\pi_i^2(x_i)}\right] (x - x_i),$$

interpolates the function $f: X \to X$, while y' interpolates f' at $x_1, x_2, \ldots, x_n$.

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